*Journal of Statistical Physics, Vol. 23, No. 6, 1980*

# Equal-Time Second-Order Moments of a Harmonic Oscillator with Stochastic Frequency and Driving Force

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*Received December 17, 1979*

Using a simple matrix method, we have obtained exact second-order equilibrium moments for a linearly damped harmonic oscillator with a fluctuating frequency $\omega(t)$ and driven by a fluctuating force $f(t)$. We have assumed each of the fluctuating quantities to be delta-correlated. We demonstrate that the final answers are identical whether $f(t)$ and $\omega(t)$ are statistically independent or delta-correlated. We have also established the region of parameter space in which the oscillator is energetically stable. The results are shown to be completely determined by the coefficients of the first and second cumulants of the fluctuations.

**KEY WORDS**: Harmonic oscillator; stochastic frequency; stochastic differential equation; stability.

## 1. INTRODUCTION

The generic problem of a linear oscillator driven by an additive fluctuating force and having a fluctuating frequency has received considerable attention recently.[1–6] Physical systems whose dynamics can be modeled by such an oscillator include spins precessing in a fluctuating magnetic field,[2] waves propagating through a random medium,[7,8] and low-amplitude wind-driven waves on the ocean surface.[9]

Van Kampen[4] has shown that one can construct a variety of linear oscillators with fluctuating parameters. Each of these oscillators has distinct properties and care must be exercised in associating a given oscillator with a given physical system. In this paper we restrict our attention to a mechanical

Supported in part by the Office of Naval Research, NSF Grant #CHE 78-21460, and by a grant from Charles and Renée Taubman.

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0022-4715/80/1200-0755$03.00/0 © 1980 Plenum Publishing Corporation

oscillator whose equation of motion for the displacement $x(t)$ is[3,6]

$$\ddot{x}(t) + 2\lambda\dot{x}(t) + \omega^2(t)x(t) = f(t) \tag{1.1}$$

where $\lambda$ is a damping parameter. The random force $f(t)$ is zero-centered and delta-correlated, i.e.,

$$\overline{f(t)} = 0 \tag{1.2a}$$

$$\overline{f(t)f(t')} = 2\tilde{D}\,\delta(t - t') \tag{1.2b}$$

The bars in (1.2) denote averages over an ensemble of realizations of $f(t)$. It will become clear later that the higher moments of $f(t)$ need not be specified. The fluctuating frequency $\omega(t)$ is also assumed to be delta-correlated. In most physical systems it seems reasonable to assume that the correlation of $\omega(t)$ and $f(t')$ is of one of the following two types. If the sources of the frequency and force fluctuations are physically independent, then $\omega(t)$ and $f(t')$ are uncorrelated for all times $t$ and $t'$. If the fluctuations have a common physical source, then $\omega(t)$ and $f(t')$ are delta-correlated. In either case we express the fluctuating frequency as

$$\begin{aligned}\omega^2(t) &= [\omega_0 + \delta\omega(t)]^2\\ &= \langle[\omega_0 + \delta\omega(t)]^2\rangle + \{[\omega_0 + \delta\omega(t)]^2 - \langle[\omega_0 + \delta\omega(t)]^2\rangle\}\\ &\equiv {\Omega_0}^2 + \gamma(t)\end{aligned} \tag{1.3}$$

where the angular brackets indicate an ensemble average over the realizations of $\gamma(t)$. Note that the brackets denote an average over the same ensemble as the bar in (1.2) when the source of the fluctuations in $\omega(t)$ and $f(t)$ is the same. In Eq. (1.3), ${\Omega_0}^2 \equiv \langle[\omega_0 + \delta\omega(t)]^2\rangle$ is defined such that the distribution of the fluctuations in the square of the frequency has zero mean, i.e.,

$$\langle\gamma(t)\rangle = 0 \tag{1.4a}$$

The assumed delta correlation of the frequency fluctuations is expressed by the cumulant relations

$$\langle\langle\gamma(t_1)\gamma(t_2)\cdots\gamma(t_n)\rangle\rangle = 2^n D_n \delta(t_1 - t_2)\cdots\delta(t_{n-1} - t_n) \tag{1.4b}$$

Note that the distribution of $\gamma(t)$ *cannot* be Gaussian if one insists that the oscillator frequency be real at all times.

In Section 2 we introduce the equations of motion for the second-order moments of the oscillator position and momentum, and obtain their equilibrium solution. Section 3 contains an analysis of the stability properties of the second moments as a function of the parameter values. The results are summarized and discussed in Section 4.

## 2. MATRIX EQUATIONS AND EQUILIBRIUM PROPERTIES

We rewrite the oscillator equation (1.1) as the set of two first-order equations

$$\dot{x} = p \tag{2.1a}$$

$$\dot{p} = -\Omega_0{}^2 x - \gamma(t)x - 2\lambda p + f(t) \tag{2.1b}$$

The properties of the solution of the stochastic differential equations (2.1) are given by the moments of the distribution of $x(t)$ and $p(t)$. Here we restrict our analysis to the equal-time second-order statistics $\langle \overline{x^2(t)} \rangle$, $\langle \overline{p^2(t)} \rangle$, and $\langle \overline{x(t)p(t)} \rangle$ in the case where $f(t)$ and $\gamma(t)$ are statistically independent. In the Appendix we show that all of our results remain unchanged when $\gamma(t)$ and $f(t)$ are delta-correlated.

We begin by constructing the equation of evolution for the column matrix $\mathbf{Y}(t)$ defined by

$$\mathbf{Y}(t) \equiv \begin{pmatrix} \frac{1}{2}\overline{x^2(t)} \\ \frac{1}{2}\overline{p^2(t)} \\ \overline{x(t)p(t)} \end{pmatrix} \tag{2.2}$$

The equation of evolution for $\mathbf{Y}(t)$ is

$$\dot{\mathbf{Y}}(t) = -[\mathbf{Z}_0 + \gamma(t)\mathbf{Z}_1]\mathbf{Y}(t) + \mathbf{d} \tag{2.3}$$

where the elements of $\mathbf{Z}_0$, $\mathbf{Z}_1$, and $\mathbf{d}$ are obtained by appropriate multiplication of Eqs. (2.1a) and (2.1b) by $x(t)$ and $p(t)$ followed by an average over an ensemble of realizations of $f(t)$. The resulting matrices are

$$\mathbf{Z}_0 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 4\lambda & \Omega_0{}^2 \\ 2\Omega_0{}^2 & -2 & 2\lambda \end{pmatrix} \tag{2.3a}$$

$$\mathbf{Z}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 0 \end{pmatrix} \tag{2.3b}$$

$$\mathbf{d} = \begin{pmatrix} 0 \\ \overline{p(t)f(t)} \\ \overline{x(t)f(t)} \end{pmatrix} \tag{2.3c}$$

To proceed further, we must perform the averages indicated in the inhomogeneous term (2.3c). From (2.1a) and (2.1b) we write, with $\Delta t > 0$,

$$\overline{x(t)f(t)} - \overline{x(t-\Delta t)f(t)} = \overline{p(t-\Delta t)f(t)}\,\Delta t \tag{2.4}$$

and

$$\begin{aligned}\overline{p(t)f(t)} - \overline{p(t-\Delta t)f(t)} = & -\Omega_0{}^2\overline{x(t-\Delta t)f(t)}\,\Delta t - \int_{t-\Delta t}^{t} dt'\,\overline{x(t')\gamma(t')f(t)} \\ & - 2\lambda\overline{p(t-\Delta t)f(t)}\,\Delta t + \int_{t-\Delta t}^{t} dr'\,\overline{f(t')f(t)}\end{aligned} \tag{2.5}$$

For delta-correlated fluctuations $f(t)$, it follows from causality that

$$\overline{x(t-\Delta t)f(t)} = \overline{p(t-\Delta t)f(t)} = 0 \tag{2.6}$$

From (2.4) it then follows that

$$\overline{x(t)f(t)} = 0 \tag{2.7}$$

Equation (2.5) then reduces in the same limit to

$$\overline{p(t)f(t)} = \lim_{\Delta t\to 0}\int_{t-\Delta t}^{t} dt'\,\overline{f(t')f(t)} = \tilde{D} \tag{2.8}$$

where we have used the statistical independence of $\gamma(t)$ and $f(t)$. Thus the inhomogeneous term in Eq. (2.3) becomes

$$\mathbf{d} = \begin{pmatrix} 0 \\ \tilde{D} \\ 0 \end{pmatrix} \tag{2.9}$$

To obtain an explicit solution for (2.3) it is convenient to introduce the interaction representation

$$\mathbf{y}(t) \equiv [\exp(\mathbf{Z}_0 t)]\mathbf{Y}(t) \tag{2.10}$$

The equation of evolution in this new variable is

$$\dot{\mathbf{y}}(t) = -\gamma(t)\mathbf{z}(t)\mathbf{y}(t) + \mathscr{D}(t) \tag{2.11}$$

where

$$\mathbf{z}(t) = [\exp(\mathbf{Z}_0 t)]\mathbf{Z}_1 \exp(-\mathbf{Z}_0 t), \qquad \mathscr{D}(t) = [\exp(\mathbf{Z}_0 t)]\mathbf{d} \tag{2.12}$$

The solution of (2.11) averaged over an ensemble of frequency fluctuations is

$$\langle \mathbf{y}(t) \rangle = \left\langle T \exp\left[ -\int_0^t \mathrm{d}\tau\, \gamma(\tau)\mathbf{z}(\tau) \right] \right\rangle \mathbf{y}(0)$$

$$+ \int_0^t dt_1 \left\langle T \exp\left[ -\int_{t_1}^t d\tau\, \gamma(\tau)\mathbf{z}(\tau) \right] \right\rangle \mathscr{D}(t_1) \qquad (2.13)$$

where we have introduced the time ordering operator $T$.

The averages occurring in (2.13) can be reexpressed as an exponential that contains sums over cumulants of $\gamma(\tau)$ of the form[2,4,5]

$$\langle\langle \gamma(\tau_n) \cdots \gamma(\tau_1) \rangle\rangle \mathbf{z}(\tau_n) \cdots \mathbf{z}(\tau_1)$$

Since the frequency fluctuations are assumed to be delta-correlated, the cumulants vanish unless the time arguments are all equal. It then follows from Eqs. (2.12) and (2.3b) that for equal time arguments

$$\mathbf{z}^n(\tau) = 0 \qquad \text{for} \quad n \geqslant 3 \qquad (2.14)$$

This is an important observation, since now all contributions beyond the second cumulant vanish and Eq. (2.13) simplifies to

$$\langle \mathbf{y}(t) \rangle = T \exp\left[ D \int_0^t d\tau\, \mathbf{z}^2(\tau) \right] \mathbf{y}(0)$$

$$+ \int_0^t dt_1\, T \exp\left[ D \int_{t_1}^t d\tau\, \mathbf{z}^2(\tau) \right] \mathscr{D}(t_1) \qquad (2.15)$$

where $D \equiv D_2$.

Rather than taking the limit $t \to \infty$ of (2.15) to obtain the equilibrium averages, it is simpler to construct the transport equation

$$\frac{d}{dt} \langle \mathbf{y}(t) \rangle = D\mathbf{z}^2(t) \langle \mathbf{y}(t) \rangle + \mathscr{D}(t) \qquad (2.16)$$

by taking the derivative of (2.15). If the oscillator is stable (cf. Section 3), then the equilibrium matrix $\langle \mathbf{Y} \rangle_{\text{eq}} \equiv \lim_{t \to \infty} \langle \mathbf{Y}(t) \rangle$ can be found by setting [see Eq. (2.10)]

$$\frac{d}{dt} \langle \mathbf{Y}(t) \rangle = \frac{d}{dt} \langle \mathbf{y}(t) \rangle - \mathbf{Z}_0 \langle \mathbf{y}(t) \rangle = \mathbf{0} \qquad (2.17)$$

Using (2.17) in (2.16) and transforming back to the original representation gives

$$D\mathbf{Z}_1{}^2 \langle \mathbf{Y} \rangle_{\text{eq}} + \mathbf{d} - \mathbf{Z}_0 \langle \mathbf{Y} \rangle_{\text{eq}} = \mathbf{0} \qquad (2.18)$$

whose solution is

$$\langle \mathbf{Y} \rangle_{eq} = (\mathbf{Z}_0 - D\mathbf{Z}_1^2)^{-1}\mathbf{d} = \frac{\tilde{D}}{2(2\lambda\Omega_0^2 - D)} \begin{pmatrix} 1 \\ \Omega_0^2 \\ 0 \end{pmatrix} \tag{2.19}$$

The equilibrium second-order moments of the oscillator displacement and momentum are then

$$\langle \overline{x^2} \rangle_{eq} = \tilde{D}/(2\lambda\Omega_0^2 - D) \tag{2.20a}$$

$$\langle \overline{p^2} \rangle_{eq} = \Omega_0^2 \langle \overline{x^2} \rangle_{eq} \tag{2.20b}$$

$$\langle \overline{xp} \rangle_{eq} = 0 \tag{2.20c}$$

Bourret *et al.*(3) have considered the same problem with frequency fluctuations modeled as a two-valued Markov process with a finite correlation time. Their results reduce to (2.20) in the limit that their correlation function approaches a delta function. Our procedure is technically simpler than theirs for delta-correlated fluctuations and clearly shows that in the case of delta-correlated fluctuations in the force $f(t)$ and frequency $\omega(t)$ the results are *completely* determined by the first- and second-order statistical properties of the force fluctuations $f(t)$ and the coefficients of the first- and second-order cumulants of the frequency fluctuations $\gamma(t)$.

We note from (2.20b) that the oscillator still maintains an asymptotic equipartition of energy, albeit in terms of the shifted frequency $\Omega_0^2$. The only other effect of the frequency fluctuations is to increase both the mean square displacement and the mean square momentum of the oscillator. This effect may be interpreted as an effective broadening of the distribution of additive fluctuations in a Wang–Uhlenbeck oscillator(10) of appropriate frequency.[3]

## 3. STABILITY OF SECOND MOMENTS

The oscillator (1.1) is referred to as "energetically stable" if all the quadratic moments relax to zero asymptotically in the absence of a driving term.(3) The time evolution of the quadratic moments when $f(t) = 0$ is found by transforming (2.16) back to the original representation and setting $\mathbf{d} = \mathbf{0}$, i.e.,

$$\langle \dot{\mathbf{Y}}(t) \rangle = (D\mathbf{Z}_1^2 - \mathbf{Z}_0)\langle \mathbf{Y}(t) \rangle \tag{3.1}$$

The solution of (3.1) is

$$\langle \mathbf{Y}(t) \rangle = \{\exp[(D\mathbf{Z}_1^2 - \mathbf{Z}_0)t]\}\mathbf{Y}(0) \tag{3.2}$$

As $t \to \infty$, $\langle \mathbf{Y}(t) \rangle \to \mathbf{0}$ provided the eigenvalues of the matrix in the

[3] For a more detailed interpretation see Lindenberg *et al.*(11)

exponent of (3.2) have negative real parts. The eigenvalue equation is readily obtained from (2.3a) and (2.3b) to be

$$\mathrm{Det}[D\mathbf{Z}_1{}^2 - \mathbf{Z}_0 - \epsilon\mathbf{I}] = \epsilon^3 + 6\lambda\epsilon^2 + 4(\Omega_0{}^2 + 2\lambda^2)\epsilon - 4(D - 2\lambda\Omega_0{}^2) = 0 \tag{3.3}$$

where $\epsilon$ is the eigenvalue and $\mathbf{I}$ is the $3 \times 3$ unit matrix. The roots of (3.3) are

$$\epsilon_1 = A + B - 2\lambda \tag{3.4a}$$

$$\epsilon_2 = -2\lambda - (A + B)/2 + i\sqrt{3}(A - B)/2 \tag{3.4b}$$

$$\epsilon_3 = -2\lambda - (A + B)/2 - i\sqrt{3}(A - B)/2 \tag{3.4c}$$

where

$$A = \{2D + [4D^2 + (\tfrac{4}{3}\omega_1{}^2)^3]^{1/2}\}^{1/3} \tag{3.5a}$$

$$B = \{2D - [4D^2 + (\tfrac{4}{3}\omega_1{}^2)^3]^{1/2}\}^{1/3} \tag{3.5b}$$

with $\omega_1{}^2 \equiv \Omega_0{}^2 - \lambda^2$.

To obtain the sign of the real parts of the roots (3.4), two cases must be distinguished.

### 3.1. Underdamped Oscillator

If the parameter values are such that

$$4D^2 + (\tfrac{4}{3}\omega_1{}^2)^3 > 0 \tag{3.6}$$

then both $A$ and $B$ can be chosen to be real. It then follows from (3.4) and (3.5) that $\epsilon_1$ is real and $\epsilon_2$ and $\epsilon_3$ are complex conjugates. Since we can conclude from (3.5) that $A > 0$, $B < 0$, and $|A| > |B|$, we immediately have from (3.4b) and (3.4c) that

$$\mathrm{Re}\,\epsilon_2 = \mathrm{Re}\,\epsilon_3 = -2\lambda - (A + B)/2 < 0 \tag{3.7}$$

for all values of the parameters. The sign of $\epsilon_1$ cannot be determined as readily from (3.4a). To obtain this sign, we note that the constant term in (3.3) is related to the product of the roots by

$$4(D - 2\lambda\Omega_0{}^2) = \epsilon_1\epsilon_2\epsilon_3 = \epsilon_1|\epsilon_2|^2 \tag{3.8}$$

Hence,

$$\epsilon_1 < 0 \qquad \text{iff} \quad 2\lambda > D/\Omega_0{}^2 \tag{3.9}$$

This relation between the damping coefficient $2\lambda$ and the ratio $D/\Omega_0{}^2$ is thus the condition for energetic stability of the underdamped oscillator.

The interpretation of (3.6) as the underdamping condition is very interesting. In the absence of the frequency fluctuations $\gamma(t)$, the oscillator is underdamped when $\lambda < \Omega_0$. Condition (3.6) modifies this requirement. The system retains its oscillatory character even for values of the damping coefficient $\lambda > \Omega_0$, provided (3.6) is satisfied. The frequency fluctuations can thus be thought of as *decreasing* the "effective" damping of the *transients* of the second moments of the oscillator. However, we have shown elsewhere[6] that the damping of the *equilibrium* correlation functions $\overline{\langle x(t)x(t+\tau)\rangle}$ and $\overline{\langle p(t)p(t+\tau)\rangle}$ is *not* modified by the frequency fluctuations.

### 3.2. Overdamped Oscillator

In this case the parameter values are such that

$$4D^2 + (\tfrac{4}{3}\omega_1{}^2)^3 < 0 \tag{3.10}$$

and the quantities $A$ and $B$ of Eq. (3.5) are complex. All the roots (3.4) can be shown by standard methods to be real. An analysis of $A$ and $B$ expressed in terms of an amplitude and a phase ($A \equiv re^{i\theta} = B^*$) leads to the conclusion that two of the three roots are negative for all parameter values and the third is negative provided

$$2\lambda > D/\Omega_0{}^2 \tag{3.11}$$

This inequality is thus again the condition for energetic stability.

The analysis for critical damping $[4D^2 + (\tfrac{4}{3}\omega_1{}^2)^3 = 0]$ obviously also leads to condition (3.11) for stability.

We finally note that since the stability analysis is carried out in the absence of the stochastic driving force $f(t)$, the results do not depend on whether $f(t)$ and the frequency fluctuations $\gamma(t)$ are or are not correlated.

## 4. CONCLUSIONS

We have presented a simple method for calculating the first and second moments of the Wang–Uhlenbeck oscillator[10] with a fluctuating frequency. Our results were obtained for delta-correlated fluctuations. The main conclusions from our analysis are:

1. The equilibrium second-order moments of the oscillator displacement and momentum depend only on the first- and second-order statistical properties of the force fluctuations. This is true even when the force fluctuations $f(t)$ have a finite correlation time.

2. The equilibrium second-order moments of the oscillator depend only on the coefficient of the second cumulant of the frequency fluctuations $\gamma(t)$ and are independent of the higher cumulant coefficients.

3. The equilibrium second-order moments, although still maintaining equipartition of energy, are *increased* in magnitude by the presence of the

frequency fluctuations. The ratio of the second-order moments in the absence and presence of the frequency fluctuations, i.e., $\langle\overline{x^2}\rangle_{D=0}/\langle\overline{x^2}\rangle_{D\neq 0}$, is $1 - D/2\lambda\Omega_0{}^2$. This result was also obtained by Bourret *et al.*[3] for a *specific* statistical model of the fluctuating frequency in the limit that these fluctuations become delta-correlated.

4. The oscillator is energetically stable provided the damping is sufficiently strong. The specific condition for stability is given by $2\lambda > D/\Omega_0{}^2$. One way to interpret this condition is that the frequency fluctuations add energy to the oscillator with rate constant $D/\Omega_0{}^2$. In order for the oscillator to be energetically stable, the dissipation rate must be sufficiently high to remove energy more rapidly than it is being added.[11] Bourret *et al.*[3] have also found this to be the condition for stability for their specific process in the delta-correlated limit.

## APPENDIX

We show that the results obtained in Section 2 remain unchanged when the fluctuations $f(t)$ and $\gamma(t)$ are delta-correlated. We begin by defining a two-component stochastic column vector $\mathbf{w}(t)$ by

$$\mathbf{w}(t) = \begin{pmatrix} x(t) \\ p(t) \end{pmatrix} \tag{A.1}$$

The equations of motion (2.1a) and (2.1b) can be expressed in terms of this vector as

$$\dot{\mathbf{w}}(t) = -[\mathbf{W}_0 + \gamma(t)\mathbf{W}_1]\mathbf{w}(t) + f(t)\mathbf{W}_2 \tag{A.2}$$

where

$$\mathbf{W}_0 = \begin{pmatrix} 0 & -1 \\ \Omega_0{}^2 & 2\lambda \end{pmatrix}, \quad \mathbf{W}_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{W}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{A.3}$$

A formal time integration of (A.2) yields

$$\mathbf{w}(t) = [\exp(-\mathbf{W}_0 t)] \int_0^t dt_1 \, T\left\{\exp\left[-\int_{t_1}^t \gamma(\tau)\tilde{\mathbf{W}}_1(\tau)\right]\right\} \times [\exp(\mathbf{W}_0 t_1)] f(t_1)\mathbf{W}_2 \tag{A.4}$$

where we have taken $\mathbf{w}(0) = \mathbf{0}$ since we are interested only in the long-time behavior of the solution. In Eq. (A.4), $T$ denotes forward time ordering and $\tilde{\mathbf{W}}_1(t)$ is the matrix $\mathbf{W}_1$ in the interaction representation, i.e.,

$$\tilde{\mathbf{W}}_1(t) = \exp(\mathbf{W}_0 t)\, \mathbf{W}_1 \exp(-\mathbf{W}_0 t) \tag{A.5}$$

The second-order moments $\langle x^2(t)\rangle$, $\langle p^2(t)\rangle$, and $\langle x(t)p(t)\rangle$ can be found by constructing the direct product

$$\mathbf{w}(t)\otimes\mathbf{w}^+(t)=\begin{pmatrix}x(t)\\ p(t)\end{pmatrix}(x(t)p(t))=\begin{pmatrix}x^2(t) & x(t)p(t)\\ p(t)x(t) & P^2(t)\end{pmatrix}\tag{A.6}$$

and averaging over the fluctuations. From (A.4) we have

$$\begin{aligned}\langle\mathbf{w}(t)\otimes\mathbf{w}^+(t)\rangle = &\left\{[\exp(-\mathbf{W}_0t)]\int_0^t dt_1\left\{T\exp\left[-\int_{t_1}^t\gamma(\tau)\tilde{\mathbf{W}}_1(\tau)\right]\right\}\right.\\ &\left.\times[\exp(\mathbf{W}_0t_1)]f(t_1)\mathbf{W}_2\right\}\\ &\otimes\left\{\mathbf{W}_2^+\int_0^t dt_2\, f(t_2)[\exp(\mathbf{W}_0^+t_2)]\right.\\ &\left.\times\left\{\overleftarrow{T}\exp\left[-\int_{t_1}^t\gamma(\tau)\tilde{\mathbf{W}}_1^+(\tau)\right]\right\}[\exp(-\mathbf{W}_0^+t)]\right\}\end{aligned}\tag{A.7}$$

where $\overleftarrow{T}$ denotes backward time ordering.

All we wish to show here is that a delta-correlation of $\gamma(t)$ and $f(t')$ does not affect the results of Section 2. To do this, all we need to show is that terms involving averages of products of $\gamma$ and $f$ do not contribute to (A.7). When the time-ordered exponentials in (A.7) are expanded, there occur terms of the form

$$\begin{aligned}&[\tilde{\mathbf{W}}_1(\tau_n)\cdots\tilde{\mathbf{W}}_1(\tau_1)\,[\exp(\mathbf{W}_0t_1)]\mathbf{W}_2]\\ &\quad\otimes[\mathbf{W}_2^+[\exp(\mathbf{W}_0^+t_2)]\tilde{\mathbf{W}}_1(\tau_m')\cdots\tilde{\mathbf{W}}_1(\tau_1')]\\ &\quad\times\langle\gamma(\tau_n)\cdots\gamma(\tau_1)f(t_1)f(t_2)\gamma(\tau_1')\cdots\gamma(\tau_m')\rangle\end{aligned}\tag{A.8}$$

The average $\langle\gamma(\tau_n)\cdots\gamma(\tau_m')\rangle$ in (A.8) can be expressed in terms of sums of products of cumulants. Two types of cumulants occur. In one type, $f$ and $\gamma$ occur in separate cumulants, e.g.,

$$\langle\langle f(t_1)f(t_2)\rangle\rangle\langle\langle\gamma(\tau_n)\cdots\gamma(\tau_m')\rangle\rangle$$

Such terms also occur when $f$ and $\gamma$ are not correlated and are therefore included in the results of Section 2. The other types of cumulants that occur contain both $f$ and $\gamma$, e.g., $\langle\langle\gamma(\tau_1)f(t_1)f(t_2)\rangle\rangle$ and various higher order terms. These are the terms that do not occur when $f$ and $\gamma$ are statistically independent, and we will show that for delta-correlated $f$ and $\gamma$, these terms still do not contribute to (A.7). To see this let us take the simplest term of this kind, namely, the one that only involves the cumulant $\langle\langle\gamma(\tau_1)f(t_1)f(t_2)\rangle\rangle$. Because the delta-correlation of $\gamma$ and $f$ implies that we must have $\tau_1 = t_1 = t_2$, the matrix product in (A.8) corresponding to this term involves

$$\mathbf{W}_1[\mathbf{W}_2 \otimes \mathbf{W}_2^+] = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{0} \tag{A.9}$$

i.e., the contribution vanishes due to the symmetry of $\mathbf{W}_1$ and $\mathbf{W}_2$. It is now easy to see that this property and the time-ordering restriction will cause *every* term that involves mixed cumulants to vanish.

## NOTE ADDED IN PROOF

Some of the results found in this paper have been obtained earlier for Gaussian delta-correlated fluctuations. Relevant references include T. K. Caughey, *J. Acoust. Soc. Am.* **32**:1356 (1960); T. K. Caughey and J. K. Dienes, *J. Math. and Phys.* **41**:300 (1962); M. A. Leibowitz, *J. Math. Phys.* **4**:852 (1963).

## REFERENCES


1. U. Frisch, in *Probabilistic Methods in Applied Mathematics*, A. T. Bharucha-Reid, ed. (Academic Press, New York, 1968), Vol. 1, p. 75.
2. R. Kubo, in *Stochastic Processes in Chemical Physics*, K. E. Shuler, ed. (Wiley, New York, 1969), p. 101.
3. R. C. Bourret, U. Frisch, and A. Pouquet, *Physica* **65**:303 (1973).
4. N. G. van Kampen, *Phys. Rep.* **24**:171 (1976).
5. R. F. Fox, *Phys. Rep.* **48**:179 (1978).
6. B. J. West, K. Lindenberg, and V. Seshadri, *Physica*, to be published.
7. V. I. Klyatskin and V. I. Tatarskii, *Sov. Phys. Usp.* **16**:494 (1974).
8. R. Dashen, *J. Math. Phys.* **20**:894 (1979).
9. O. M. Phillips, *J. Fluid Mech.* **2**:417 (1957).
10. M. C. Wang and G. E. Uhlenbeck, *Rev. Mod. Phys.* **17**:323 (1945).
11. K. Lindenberg, V. Seshadri, and B. J. West, *Phys. Rev. A*, to be published.